Dr. James Girard Summer Undergraduate Research Program
Faculty Mentor – Project Application

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Achievement and Avoidance Games Played on Graphs

Abstract

This research project studies impartial games played on graphs and contributes to the field of combinatorial game theory. An impartial game is a 2-player game in which the possible moves are the same for each player in any position. Every impartial game consists of a finite set of positions and a collection of options for each position. In every move of the game, a player chooses a new position from the options of the current position. No position can be visited twice. Each position is associated with a nonnegative value, called the nim number. The nim number of a position determines whether the position is winning (nim value 0) or losing (positive nim value) for the player moving into that position. Therefore, the winning strategy for any impartial game is to always move to an option with nim number 0, if available.

In this research, we study impartial games played on connected graphs in which players take turn selecting previously unselected vertices of a graph until certain conditions are met. In the achievement game, the first player who chooses the vertex such that the selected vertex set generates all the vertices of the graph wins the game. In the corresponding avoidance variation, the player who cannot select a vertex without generating all vertices of the graph loses. This research initiates aPnew combinatorial(i)-2 (t)-2 (h ni)-2 (m)-2 2 (i)0.4f(nc)9 (o)-4rng Thiin wt p -2 (ng p)]TJon, /

(à) = 0. The winning strategy is to always move to an option with nim number 0 if available. For an impartial game, a nim number of 0 denotes the second player has a winning strategy for the game whereas a positive nim number denotes the first player has a winning strategy for the game. In addition to determining game outcomes, the nim number makes it easy to compute the nim number of sums of games.

Introduction

This project studies impartial games involving geodesics on graphs. A *graph* consists of a finite vertex set *V* and a set of 2-element subsets of *V* called an edge set *E*. We require graphs to be connected, that is, there exists a path between any pair of vertices. A *geodesic* is any path of minimum length between two vertices in a graph. A *vertex-set geodesic* is the minimum length geodesic between the vertex and any vertex in the set. For a subset of vertices *S*, we define the subset *v-S* to be the union of *S* and the vertices in the vertex-set geodesic between *v* and *S*.

We study two game variations in this research project: achievement and avoidance. In both games, players take turns selecting unselected vertices of a graph until the vertex-set geodesic of selected vertices becomes too large. The last player to move is the winner. The achievement game ends when the final vertex-set geodesic, *v-S*, contains every vertex. In other words, the player who generates the whole vertex set first wins. In the avoidance game, the vertex-set geodesic is not allowed to contain every vertex in the graph. The avoidance game ends when there are no available moves. The goal of this project is to solve these games by determining winning strategies and nim numbers for various graph families and graph constructions. We prove our results according to graph family. While there are some known results, there are many cases yet to be considered.

Other geodetic games have been studied in [3,5,6,8,9,11,15]. These previous games relied on the following definitions. A subset of the vertex set of a graph is *geodetically convex* if it contains every vertex on any shortest path between two elements of the set. The *convex hull* of a set of vertices is the smallest convex set containing the set. Then the *geodetic closure* of a subset of vertices as the convex hull of the set of geodesics. In the earlier versions of geodetic games, the the geodetic closure was used to define the game rules. Buckley and Harary studied the geodetic closure achievement game GEN in cycles, complete graphs, wheels (and generalized wheels), complete bipartite graphs, hypercubes, and the Petersen graph [6]. For the associated avoidance game DNG, they studied cycles, wheels (and generalized wheels), complete bipartite graphs, and complete graphs. Haynes studied both the achievement and avoidance games GEN and DNG for complete multipartite graphs, coronas, complete block graphs, and split graphs [9]. In many cases, both groups studied particular critical configurations for positions that determine the end of the game. The games presented here are a generalization of the game in [7,10]. My collaborators and I introduced the convex hull variation of these games in [3]. Consequently, we produce different results from these adjusted definitions. In this collaboration, we solved four game variations in the following graph families: trees, cycles, complete, complete bipartite, windmill, lattice, wheel, and multipartite graphs. In this research project, we will apply similar techniques from these completed previous works in order to increase the potential for success in this project.

Approach/Methodology

Nim numbers can be computed through the use of Sage, which is available in an open source via CoCalc. My research collaborators and I have developed code to generate nim

Proposed Timeline that includes Aims and/or Goals

Week	Action	Goal for Student
1	Assign student to read selected chapters from Reference list for background on this research. Give student a list of corresponding exercises to solve while working through the readings. Have a daily check-in during the first week to answer questions and discuss readings.	Understand and solve problems using the basics of combinatorial game theory, nim numbers, mex algebra, graphs, and geodetic closure
2	Introduce the rules for the achievement game and avoidance game as well as solved game examples. Guide student to solve games on basic families of graphs: paths, cycles, complete, and complete bipartite.	Learn how to generalize results from experimental games on select graphs
3	Introduce maximal nongenerating sets as a way to solve game and proof techniques to prove nim number results. Have student draft proofs of results from basic graphs. Encourage student to experiment with new families of graphs and/or graph constructions based on interest.	Use and adapt existing software to compute nim values for experimental graphs
4	Take all results up to this point and write complete proofs using LaTeX. Student will submit drafts to get feedback several times throughout this week.	Become familiar with the process of proof writing and what makes a concise proof
5	Introduce the rules for the achievement game and avoidance game as well as solved game examples. Guide student to solve games on basic families of graphs: trees, cycles, complete	With 4 game variations, student can now take full ownership of project and determine future directions
6-8	Guide student to experiment with new cases and ask productive research questions. Introduce structure theory if nim number computations become too complicated to prove with existing techniques.	Develop curiosity and learn how to ask quality research questions to further results
9-10	Finalize results, write upW nBT12 -0 0 12 402.36 332.76	

References

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[5] Fred Buckley and Frank Harary. Closed geodetic games for graphs. Combinatorics, graph theory and computing, Proc. 16th Southeast. Conf., Boca Raton/Fla. 1985, Congr. Numerantium

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